Group : (Def^n)

A group is a non empty set G of elements together with a binary operation "o" satisfying the following Postulates is called a group

If $a, b, c \in G$ then

(i) $a o b \in G$ (Closure law

(ii) The operation is associative

$$a o (b o c) = (a o b) o c$$

(iii) There exist an element $e \in G$ such that

$$a o e = e o a = a.$$

The element e is known as an identity element in G

(IV) ~~For every $a \in G$, there exists an element $b \in G$ such that~~

Existence of inverse.

Each element of G possesses inverse, In otherwords $a \in G \Rightarrow$ there exist an element $b \in G$ such that

$$b o a = e = a o b$$

The element b is then called the inverse of a and we write $b = a^{-1}$. Thus a^{-1} is an element of G such that a^{-1} ~~$a = e = a$~~

$$a^{-1} o a = a o a^{-1} = e$$

(Commutative)

Abelian Group :- A group (G, o) is called an abelian or Commutative if and only if 'o' is Commutative i.e $a \circ b = b \circ a \ \forall \ a, b \in G$.

Then the group G is called abelian group

Finite Group : A group (G, o) is said to be finite if the underlying set G consists of only a finite no. of elements.

Infinite group :- ~~A group (G, o)~~

If the group contains an infinite no. of elements then the group is called an infinite group.

The no. of elements in a group is called its order. An infinite group has an infinite order.

Examples of Abelian group or Commutative group

(i) The set of all even integers with zero forms an abelian group with respect to addition composition.

(ii) The set of integers including zero forms a group with respect to addition.

(iii) The set of real no. with addition as operation form a group.

QN:- If $a, b, c \in G$ then

$ab = ac \Rightarrow b = c$ [Left Cancellation law]

$ba = ca \Rightarrow b = c$ (Right Cancellation law)

or, Cancellation laws hold good in a group

Ans.→

Proof:- $a \in G \Rightarrow \exists\, a^{-1} \in G$ such that,

$a^{-1}a = e = aa^{-1}$

where, e is the identity element

Now, $ab = ac \Rightarrow a^{-1}(ab) = a^{-1}(ac)$

(multiplying both sides on the left by a^{-1}]

$\Rightarrow (a^{-1}a)b = (a^{-1}a)c$ [by associativity]

$\Rightarrow eb = ec$ [$\because a^{-1}a = e$]

$\Rightarrow b = c$ [$\because e$ is identity]

Also, $ba = ca \Rightarrow (ba)a^{-1} = (ca)a^{-1}$

$\Rightarrow b(aa^{-1}) = c(aa^{-1})$

$\Rightarrow be = ce$

$\Rightarrow b = c$.

—o—

QN.→ If a and b are elements of a group G, the equations

(i) $ax = b$

and (ii) $ya = b$

have unique solutions in G.

Ans. → (i) Consider the eqn

$$ax = b \quad \text{——} \quad (1)$$

We are going to show that $a^{-1}b$ is the solution of the given equation.

It has to be observed that $a^{-1}b \in G$, for a^{-1} and $b \in G$ and therefore $a^{-1}b \in G$.

If $a^{-1}b$ is the solution of the eqn. then, $x = a^{-1}b$ must satisfy the given eqn.

Now, Putting $x = a^{-1}b$ in (1) we have the L.H.S.

then, $a^{-1}a = e = aa^{-1}$

$= a(a^{-1}b) = (aa^{-1})b$ [by associative

$= eb$ [defn. of Inverse a^{-1}]

$= b$ [by defn of the identity]

Therefore the eqn. has a solution $x = a^{-1}b$.

Now we are going to show that $x = a^{-1}b$ is the unique solution. If not, suppose $x = c$ is another solution in G.

Putting $x = c$ in (1), we have

$$ac = b$$

multiplying both sides by a^{-1} on the left, we have,

$$a^{-1}(ac) = a^{-1}b$$

or, $(a^{-1}a)c = a^{-1}b$

or, $ec = a^{-1}b$

$\therefore c = a^{-1}b$

Which means that whatever solution we assume for the given eqn, it will come out be $a^{-1}b$.

Hence, the Solution $x = a^{-1}b$ is unique.

(ii) Conside the eqn

$$ya = b \quad — (2)$$

We are going to show that $a^{-1}b$ is the Solution of the given equation.

It has to be observed that $a^{-1}b \in G$, for a^{-1} & $b \in G$ and therefore $a^{-1}b \in G$.

If $a^{-1}b$ is Solution of the eqn then, $y = a^{-1}b$ must satisfy the given eqn.

Now, Putting $y = a^{-1}b$ in (2), we get the R.H.S.

$= (a^{-1}b)\,a^{-1}$

$= b(a a^{-1})$ [by Associative]

$= be$ [by Defn of Inverse a^{-1}]

$= b$ [by Defn. of the identity]

Therefore the eqn has a Solution $y = a^{-1}b$

Now, we are going to show that $y = a^{-1}b$ is the unique Solution; If not, Suppose $x = c$ is another Solution in G.

Putting $x = c$ in (2), we have

$$ca = b$$

Multiplying both sides by a^{-1} on the Right we have

$$(ca)\,a^{-1} = ba^{-1}$$

$$\text{or, } c(aa^{-1}) = ba^{-1}$$

$$\text{or, } ce = ba^{-1}$$

$$c = ba^{-1}$$

Which means that whatever Solution we assume for the given eqn, it will come out be ba^{-1}

Hence, the Solution $y = ba^{-1}$ is unique.

Q.N → To State and Prove uniqueness of Inverse.

'or'

The Inverse of an element in a group is unique.

or,

Prove that in a group (G) each element a of G has a unique inverse a^{-1} in G.

Ans → Let G be a group. Let a be an element of G and let a^{-1} be its inverse.

we have to prove that a^{-1} is unique.

If not, suppose a' is another inverse of a.

Since a^{-1} is the inverse of a, therefore,

$$aa^{-1} = a^{-1}a = e \text{ ——— } ①$$

Similarly, since a' is the inverse of a, therefore,

$$aa' = a'a = e \text{ ——— } ②$$

Where e is the identity element G.

multiplying ① by a' on the left, we have

$$a'(aa^{-1}) = a'e = a' \text{ ——— } ③$$

multiplying ② by a^{-1} on the right, we have,

$$(a'a)a^{-1} = ea^{-1} = a^{-1} \text{ ——— } ④$$

Since by associative law,

$$a'(aa^{-1}) = (a'a)a^{-1}$$

therefore we have from ③ & ④

$$a' = a^{-1}$$

Hence the inverse of an element in a group is unique.

(or)

Ans→ Let a be any element of a group G and let e be the identity element. Suppose b and c are two inverses of a,

i.e. $ba = e = ab$

& $ca = e = ac$

We have,

$b(ac) = be$ [$\because ac = e$]

$= b$ [$\because e$ is identity]

Also, $(ba)c = ec$ [$\because ba = e$]

$= c$ [$\because e$ is identity]

But in a group composition is associative. Therefore,

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$b(ac) = (ba)c$

$\therefore b = c.$